Derived annular Khovanov-Rozansky invariants - part 1
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based on joint work with Eugene Gorsky, Matt Hogancamp, Dave Rose, Hoel Queffelec

Reasons to study annular link homology:

- extension to ambient manifolds beyond $\mathbb{R}^3/B^3/S^3$
- intrinsically interesting algebraic structures arise
- in some cases better behaved than in $\mathbb{R}^3/B^3/S^3$

First goal: annular link homology à la Queffelec-Rose-Sartori et.al.

- a universal annular Khovanov-Rozansky invariant
- defined using the horizontal trace construction
- categorifying the ring of symmetric functions $/\mathbb{Z}[q^{\pm}]$
- examples & open questions

Then: derived annular link homology Gorsky-Hogancamp-W.

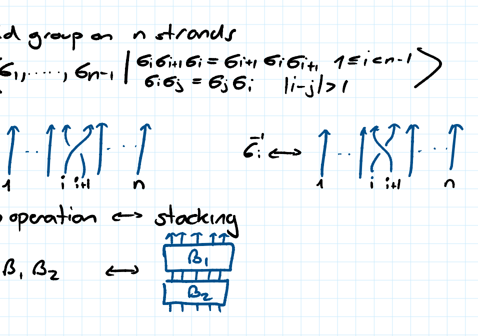
- a universal Khovanov-Rozansky invariant for braid closures in $S^1\times D^2$
- defined via a derived horizontal trace
- towards categorified skein theory

I Motivation & introduction

I.1 $Br_n :=$ braid group on n strands

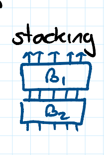
$$= \left\langle \sigma_1,\dots,\sigma_{n-1} \;\middle|\; \begin{array}{ll}\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & 1\le i\le n-1\\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1\end{array}\right\rangle$$

$\sigma_i \leftrightarrow$ $\quad\sigma_i^{-1} \leftrightarrow$



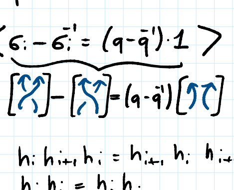
group operation $\leftrightarrow$ stacking

$\beta_1\beta_2 \leftrightarrow$



I.2 $H_n :=$ Hecke algebra of type A_{n-1}

$$= \mathbb{Z}[q^{\pm1}]Br_n \Big/ \langle \underbrace{\sigma_i - \sigma_i^{-1} = (q-q^{-1})\cdot 1}\rangle$$



rename $[\sigma_i]\mapsto h_i$

$$= \left\langle h_1,\dots,h_{n-1} \;\middle|\; \begin{array}{ll} h_ih_{i+1}h_i = h_{i+1}h_ih_{i+1} & 1\le i\le n-1\\ h_ih_j = h_jh_i & |i-j|>1 \\ (h_i+q^{-1})(h_i-q)=0 & \end{array}\right\rangle_{\mathbb{Z}[q^{\pm1}]}$$

a "finite-rank" algebraic braid invariant $\quad (q=v^{-1})$

I.3 Jones, Ocneanu: There exists a system of traces (linear map f with $f(ab)=f(ba)$)

$tr_n: H_n \to \mathbb{Q}(q)[a^{\pm1}]$ for $n\geq 0$

uniquely determined by:

$tr_0(1)=1$

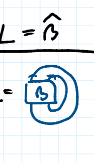
$tr_{n+1}(\boxed{\beta}\,|) = \frac{a-a^{-1}}{q-q^{-1}}\, tr_n(\boxed{\beta})$

$tr_{n+1}(\boxed{\beta}\text{ with positive crossing}) = -a^{-1}\, tr_n(\boxed{\beta})$ and similarly with $-a$ for the negative crossing

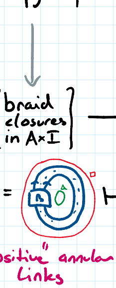
I.4. HOMFLYPT:

$$\begin{Bmatrix}\text{Links}\\ \text{in } \mathbb{R}^3\end{Bmatrix} \xrightarrow[\text{Theorem}]{\text{Alexander's}} \{\text{braids}\} \longrightarrow \bigoplus_{n\geq0} H_n \xrightarrow{\Sigma tr_n} \mathbb{Q}(q)[a^{\pm1}]$$

$L \xmapsto{L=\hat\beta} \beta \longmapsto [\beta] \longmapsto P_\infty(\hat\beta) = P_\infty(L)$ Markov

$L =$ 

$$\begin{Bmatrix}\text{braid}\\ \text{closures}\\ \text{in } A\times I\end{Bmatrix} \longrightarrow \bigoplus_{n\geq 0} H_n/[H_n,H_n] \xrightarrow[\text{Turaev}]{\cong} \Lambda_q \quad \text{some alg hom; symmetric functions } /\mathbb{Z}[q^{\pm1}]$$

$\hat\beta =$  $\longmapsto [\hat\beta] \longmapsto tr(\hat\beta) = \sum_\lambda Tr_{V_\lambda}([\beta])\, s_\lambda$

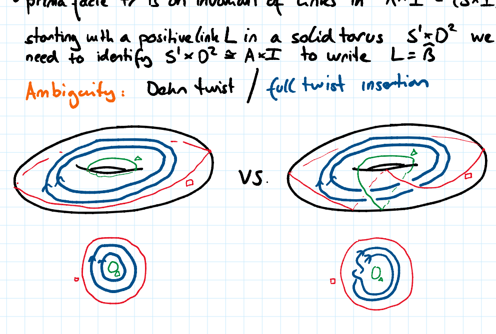
"positive" annular links — positive half of the HOMFLYPT skein of the annulus, Przytycki, Turaev

I.5 Remarks:

- prima facie tr is an invariant of links in $A\times I = (S^1\times I)\times I$

starting with a positive link L in a solid torus $S^1\times D^2$ we need to identify $S^1\times D^2 \cong A\times I$ to write $L=\hat\beta$

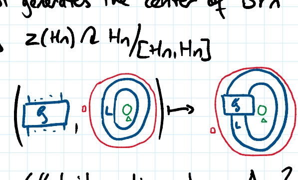
Ambiguity: Dehn twist / full twist insertion



- the full twist generates the center of Br_n

fix n & study $z(H_n) \curvearrowright H_n/[H_n,H_n]$

topologically:



- Q: how does full twist insertion act on Λ_q?

Goal: categorify all this.

II Tools pre categorification

II.1 Bases for H_n.

fix a reduced expression $\underline{w}: s_{i_1}\cdots s_{i_{\ell(w)}}$ for each $w\in S_n$

- $\{h_w = h_{\underline{w}} := h_{i_1}\cdots h_{i_{\ell(w)}} \mid w\in S_n\}$ "images of positive permutation braids"
- $\{b_w \mid w\in S_n\}$ Kazhdan-Lusztig basis

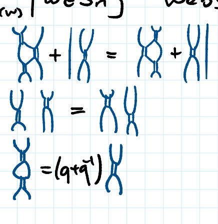
e.g. $b_i = h_i + q^{-1}1 = h_i^{-1} + q1$ — invariant under crossing change, $h_w \leftrightarrow h_{w^{-1}}^{-1}$, $q \leftrightarrow q^{-1}$

the b_i satisfy relations

$$\begin{array}{ll} b_ib_{i+1}b_i + b_{i+1} = b_{i+1}b_ib_{i+1} + b_i & 1\le i\le n-1\\ b_ib_j = b_jb_i & |i-j|>1 \\ b_i^2 = (q+q^{-1})b_i & \end{array}$$

- $\{b_{\underline{w}} := b_{i_1}\cdots b_{i_{\ell(w)}} \mid w\in S_n\}$ "webs" useful for computation, accessible for categorification

e.g.:



Q: What are $tr(b_w), tr(b_{\underline{w}}) \in \Lambda_q$?

II.2 q-Schur-Weyl duality. $V=\mathbb{C}^N \quad V_q = \mathbb{C}(q)^N$

$S_n \curvearrowright \underbrace{V\otimes\cdots\otimes V}_{n \text{ factors}} \curvearrowleft U(\mathfrak{gl}_N)$ — have double centralizer property

$H_n \curvearrowright \underbrace{V_q\otimes\cdots\otimes V_q}_{n \text{ factors}} \curvearrowleft U_q(\mathfrak{gl}_N)$

Upshot: $H_n \twoheadrightarrow \underbrace{\mathrm{End}_{U_q(\mathfrak{gl}_N)}(V_q^{\otimes n})}$ $\cong$ if $N\geq n$

This is where Reshetikhin-Turaev braid invariants for $\mathfrak{gl}_N$ live.

II.3 q-skew Howe duality

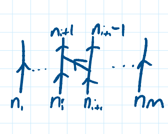
$\dot{U}_q(\mathfrak{gl}_m) \curvearrowright \Lambda^n(\mathbb{C}(q)^m \otimes V_q) \curvearrowleft U_q(\mathfrak{gl}_N)$

$\mathfrak{gl}_m$ weight $\underline{n} = (n_1,\dots,n_m) \leftrightarrow \Lambda^{n_1}V_q\otimes\cdots\otimes\Lambda^{n_m}V_q$

$$1_{\underline{e}}\dot{U}_q(\mathfrak{gl}_m)1_{\underline{n}} \overset{(*)}{\twoheadrightarrow} \underbrace{\mathrm{Hom}_{U_q(\mathfrak{gl}_N)}\Big(\bigotimes_i \Lambda^{n_i}V_q, \bigotimes_i \Lambda^{e_i}V_q\Big)}$$

this is where Λ-colored R-T invariants of braids live

$E_i1_{\underline{n}} \longmapsto$



built from: $\Lambda^aV_q\otimes\Lambda^bV_q \rightleftarrows \Lambda^{a+b}V_q$

II.4 Cautis-Kamnitzer-Morrison: the kernel of $(*)$ is generated by $1_{\underline{n}}$ with some $n_i \notin \{0,\dots,N\}$

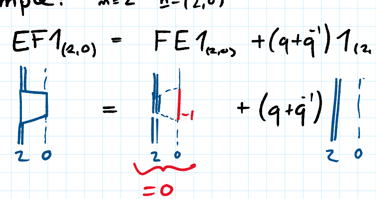
$\Rightarrow$ a presentation of $\mathrm{Fund}\, U_q(\mathfrak{gl}_N)$ as a pivotal $\otimes$-cat by diagrammatic generators & relations

Elias: over $\mathbb{Z}[q^{\pm1}]$!

This is why link homology works so well in type A.

II.5 Example: $m=2 \quad \underline{n}=(2,0)$

$EF1_{(2,0)} = FE1_{(2,0)} + (q+q^{-1})1_{(2,0)}$

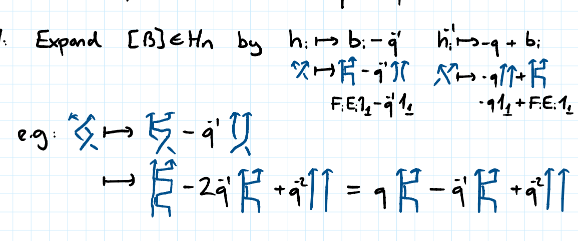


compare this with $b_i^2 = (q+q^{-1})b_i$

II.6 How to compute $tr(\beta)\in\Lambda_q$ in practice?

Step 1: Expand $[\beta]\in H_n$ by $h_i \mapsto b_i - q^{-1}$, $\quad h_i^{-1} \mapsto -q + b_i$

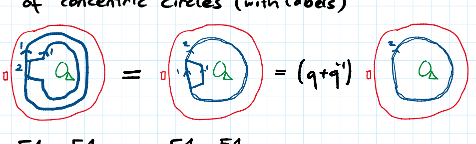
$F_iE_i1_{\underline{1}} - q^{-1}1_{\underline{1}} \qquad -q1_{\underline{1}} + F_iE_i1_{\underline{1}}$



Step 2: Queffelec-Rose "annular simplification"

Idea: simplify each annular web into a $\mathbb{Z}[q^{\pm1}]$-linear combination of concentric circles (with labels)

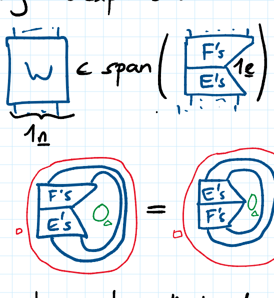
e.g.:



$F1_{(2,0)}E1_{(1,1)} \qquad E1_{(1,1)}F1_{(2,0)}$

How? Use q-skew Howe duality + PBW/Δ-decomposition for $\dot{U}_q(\mathfrak{gl}_m)$

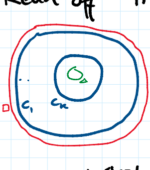
$\Rightarrow$ every web expands as:



$\underline{e} > \underline{n}$ unless W is an identity web

Now we have webs with boundary $\underline{e}>\underline{n}$. Iterate.

Step 3: Read off $tr(\beta)$ via the dictionary

 $\leftrightarrow e_{c_1}\cdots e_{c_k} \qquad e_c = s_{1^c}$ elementary symm. fctn.

eg $tr(\text{crossing}) \overset{\text{step 1}}{=} tr(q\,\text{web} - q^{-1}\,\text{web} + q^{-2}\,||)$

$\overset{\text{step 2}}{=} tr\big((q^2+1)\,|_2 - (1+q^{-2})\,|_2 + q^{-2}\,|_1|_1\big)$

$\overset{\text{step 3}}{=} q^2e_2 - q^{-2}e_2 + q^{-2}e_1^2 = q^2 s_{\square\!\square^{\text{col}}} + q^{-2}s_{\square\square}$

compare with

$tr(||) = e_1^2 = s_{\text{col}} + s_{\text{row}}$

e.g. (Gorsky-W (coloured setting)):

$tr(\text{web}) = q^{n-1}s - q^{n-3}s + \cdots \pm q^{1-n}s$

$= (-1)^{n-1}\frac{h_n[X(q^{-1}-q)]}{q^{-1}-q}$

more generally: $tr(\text{web}_{\varepsilon_1\cdots\varepsilon_{n-1}}) \sim$ plethystically transformed ribbon Schur fcn $(\underline{\varepsilon})$

II.7 Proposition (Gorsky-W): Let W be a web and $w\in S_n$. $tr(W)$ and $tr(b_w)$ are Schur positive.

Conjecture: they are indeed e-positive.

III Categorification

III.1. Soergel bimodules. Fix $n\geq 0$.

cheat sheet for bimodules

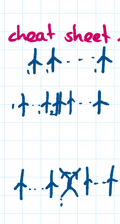
- $R := \mathbb{C}[x_1,\dots,x_n] \quad \deg(x_i):=2$

$\cup$

- $R^{S_i} :=$ invariants under $x_i \leftrightarrow x_{i+1}$

$= \mathbb{C}[x_1,\dots,x_i+x_{i+1}, x_ix_{i+1},\dots,x_n]$

- $B_i := R\otimes_{R^{S_i}} R(1)$ ← grading shift down



$SBim_n :=$ full subcat of graded R-R-bimodules

- containing $R, B_i \quad 1\le i\le n-1$
- closed under $\otimes_R, \oplus, \ominus$, grading shift.

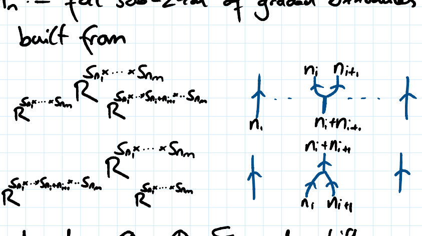
– a graded, $\mathbb{C}$-linear (not abelian), monoidal category

$K_0(SBim_n) \cong H_n$

More generally: $SSBim_n :=$ full sub-2-cat of graded bimodules "singular"

- built from

$R^{S_{n_1}\times\cdots\times S_{n_m}}$, $R^{S_{n_1}\times\cdots\times S_{n_i+n_{i+1}}\times\cdots\times S_{n_m}}$



- closed under $\otimes, \oplus, \ominus$, grading shift.

– a graded, $\mathbb{C}$-linear (not abelian) 2-category